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Analyzing the Impact of Time
Horizon, Volatility and Profit Margins on
Solvency Capital: Proposing a New Model
for the Global Regulation of the
Insurance Industry

Thomas Müller



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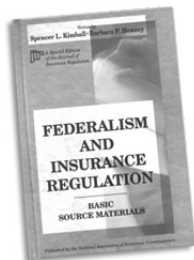
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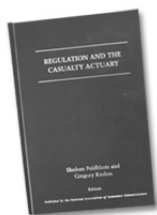
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Analyzing the Impact of Time Horizon, Volatility and Profit Margins on Solvency Capital: Proposing a New Model for the Global Regulation of the Insurance Industry

Thomas Müller *

Abstract

The European Solvency II regime requires a solvency capital covering risks with a given shortfall probability of $1/200=0.5\%$ on a one-year time horizon, which is extremely short compared to the contractual terms in traditional life insurances, as well as the settlement periods of several decades in some casualty branches. This approach undermines the importance of a high return margin and, given a risk-averse approach to management, may lead to an overall riskier business strategy in the long run. In light of this, we cannot help but ask whether such a short time horizon is capable of providing a meaningful guideline for a sustainable business and risk management that has a long-term perspective.

In response to this question, we present a new model for assessing the evolution of the equity of an insurance company and calculating the probability that the initial equity of an insurance company will be depleted during a given time period. This model demonstrates that insolvencies mostly do not occur in the first year. Therefore, if one only considers a one-year window, as is the case under Solvency II, the risk will be underestimated. Even more serious is that the business will be managed too cautiously without aiming for a suitably high profit margin, which significantly reduces the risk only in the long term.

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1. Introduction

Over the past 20 years, the debate on the appropriate solvency requirements for the insurance industry has become increasingly more global. Colliding within it are fundamentally different concepts regarding the life and non-life insurance sectors and the methods such as the rule-based or stochastic approaches, coming from the leading economies of the U.S., European Union (EU) and Asia. Critical to the definition of any solvency requirement is the length of the time horizon used for a company's financial projections. It is here that the U.S., for example, differs significantly from the EU.

Within the next few years, the International Association of Insurance Supervisors (IAIS) will be developing a global insurance capital standard (ICS) for internationally active insurance groups (IAIGs) and global systematically important insurers (G-SIIs). Currently, it appears that this global standard ICS will incorporate the key points of the Solvency II system, which requires sufficient solvency capital to cover risks with a specified probability of default of 0.5% over a one-year time horizon. In statistical terms, this would allow for an insurance company to go bankrupt once in 200 years. At least this is what the current field test for ICS version 1.0 envisages, cf. IAIS (2017). However, serious questions remain as to whether this one-year window can function as a meaningful guideline for the insurance industry.

Clearly, the length of the time horizon itself—whether it is a single year, as stipulated in the Solvency II standard formula, or longer as required by the European Own Risk and Solvency Assessment (ORSA) Guidelines, cf. European Insurance and Occupational Pension Authority (EIOPA) (2015)—sets the stage for major differences in business management. With a short time horizon, the risk itself becomes crucial for the solvency capital requirement. On the other hand, the estimated profit margin only plays a minor role. The importance of the margin greatly increases when looking at longer time horizons of several years or even decades.

As mentioned above, the Solvency II regime requires a solvency capital covering risks with a given shortfall probability of $1/200 = 0.5\%$ ¹ on a one-year time horizon. However, this one-year time horizon is very short compared to the contractual terms in traditional life insurances, as well as to the settlement periods of several decades in some casualty branches. This undermines the importance of a high return margin and, given a risk-averse business management, may lead to an overall riskier business strategy in the long run. Indeed, the question arises, if such a short time horizon can, in fact, serve as an appropriate guideline for a sustainable business and risk management strategy with a long-term perspective rather than just a potentially wrong incentive that aims at a higher, but short-term, security level.

1. Usually this is reversed by saying that Solvency II requires sufficient risk capital to ensure that at least 99.5% of companies are still solvent after one year, cf. EIOPA (2009).

In this paper, we present a new model for understanding the evolution of the equity of an insurance company and calculating the probability that the initial equity of that company will be depleted during a given time period. A key focus of our examination is the one-year window requirement, for which we consider both its implications and viability for the insurance industry. It should be pointed out that this issue mainly concerns life insurance, including variable annuities with a contract duration of decades, and has more limited implications for non-life insurance where a one-year time horizon may be adequate for risk assessment due to the typical one-year contract duration.

2. Background and Significance of Approach

In recent years, the important developments among insurance regulators around the world have been closely followed in the U.S. and are shaping the debate on the revision of the long-standing risk-based capital (RBC) standards there. However, initiatives to revise the existing rules-based system at the beginning of the millennium—Solvency Modernization Initiative (SMI)—did not lead to the repeal of the rules-based U.S. solvency standard. A brief overview of the literature on these recent developments in global solvency regulation can be found in Mao, Carson, Ostaszewski and Hao (2015). As Eling and Schmeiser (2010) and many other critical essays on this topic have shown,² the current global debate has been taking place primarily in academic circles, where there is little understanding of the rule-based U.S. approach. However, there have also been contributions in the academic literature demonstrating that the concept of principle-based supervision is not as straightforward and unambiguous to implement as the clear theoretical principles suggest. Table 6 in Eling (2012), for example, reveals large differences in the resulting risk capital, depending on which statistical method is applied to the empirical loss event data for non-life insurance.

Moreover, the U.S. RBC standard cannot simply be transferred to other markets outside the U.S. Their risk factors are based on U.S. claims experience and depend on the circumstances in the U.S., thinking only of health insurance, medical malpractice or environmental pollution and see also Hooker et al. (1995) on the catastrophic effects of judicial, legislative and regulatory decisions. By contrast, the stochastic model of Solvency II seems much better suited to a global framework that is independent of the local characteristics of the insurance risks assumed. Having said this, within this rule-based U.S. RBC standard, there is a modern stochastic model that we consider to be groundbreaking. This concerns a modern life insurance product, the variable annuities, whose capital requirements are calculated in a stochastic model that takes into account the entire period up to the expiry of the contracts.³

2. See also Eling and Holzmüller (2008) or Mao et al. (2015).

3. See Section 8 for more details.

A key point of contention in current debates is how solvency requirements can be designed in an optimal manner. In this regard, the financial literature contains a number of publications dealing directly with this issue, cf. Stoyanova and Schlütter (2015), Yow and Sherris (2008) and Myers and Read (2001). These studies usually model the risk component on two different geometric Brownian motions—one for the assets and one for the liabilities. This allows the probability of default to be calculated according to Magrabe's formula. By contrast, the model we propose does not consider separate Brownian motions for assets and liabilities, but uses only one Brownian motion—sometimes also called arithmetic Brownian motion—to differentiate it from the geometric Brownian motion, which in our model describes the motion of equity overall—i.e., the surplus of assets over liabilities or also a possible deficit.

The random path for an arithmetic Brownian motion is modeled by sums of random terms, while this is done for geometric Brownian motions by products of random factors. In contrast to the geometric Brownian motion, the arithmetic Brownian motion in our model can also move into the negative range, as can also be the case if the stochastic motion of the company's balance sheet is modeled by two separate geometric Brownian motions—one for the assets and another one for the liabilities. Geometric Brownian movements lead to lognormal distributions for the corresponding random variables, while arithmetic Brownian movements lead to the better-known normal distributions. In the Myers and Read (2001) study, the two separate lognormal distributions—one for assets and one for liabilities—were modified by only considering a normally distributed equity. As far as the stochastic model is concerned, this approach largely corresponds to what is referred to here by the shortfall probabilities in the simple model. This relationship is described in more detail in chapter 9.

The main difference between most models in the financial literature and our model is that their models almost exclusively consider the probability that there will be a negative equity at a fixed future point in time, usually after one year,⁴ which will be called the "probability of shortfall" hereafter. Their considerations require mathematical methods such as the Magrabe formula, which always take into account only a certain future point in time. This naturally raises the question of when the solvency requirements should apply: in one year, five, 10 or perhaps 20 years?

In the approach employed here, this question does not arise, or at least it is much less relevant, since we look at whole periods. For example, we wonder what the probability is that an insurer can stay in business for the next 20 years, thus the probability to survive the next 20 years. This allows us to look at much longer periods of time, which is the appropriate view for long-term insurance contracts.

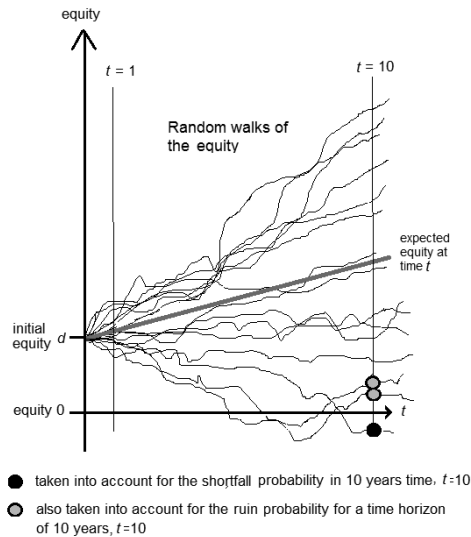
If you only want to determine the probability of shortfall in our simpler model, where the motion of the equity is modeled by an arithmetic Brownian

4. There are occasions when the fixed future point is much later, such as when the insurance contracts expire, as in the literature on the calculation of risk measures for variable annuities guarantees. This is explained in more detail in Section 9.c.

motion, no further financial techniques are required, as for instance the Magrabe formula in the studies mentioned above. In our model, equity is normally distributed at each given specific future date. However, what is significant here is that we are not really interested in the shortfall probability at a point in time, but in the probability that the equity is depleted during a whole period of time.

This is obtained by the stopping hypothesis, which states that those companies whose equity once reached the zero line and was thus exhausted can never become solvent again and are considered to be out of business. This additional assumption is natural from an economic point of view, but it catapults the mathematical problems to be solved into another dimension. In order to solve this question, usually very advanced stochastic methods are required, since all possible paths that the equity capital can take in the regarded time period must now be considered. (See Figure 1.) Therefore, we need more sophisticated stochastic models without neglecting previous insolvencies.

Figure 1



We estimate a probability of 1/15 for each of the 15 shown random walks. Then, for a one-year time horizon, the shortfall—as well as the ruin probability—is zero. Indeed, all random walks are still quite close to the initial equity. At time horizon $t=10$, the walks spread over a larger area due to the diffusion of the process described by the volatility σ . In one of the 15 cases, the equity at time $t=10$ is negative, and the simple shortfall probability only considers this case. In contrast, all the three walks undercutting the zero equity line before $t=10$ are taken into account for the probability of ruin.

time horizon= years t	1	10
Probability of shortfall, with-out stopping when the equity has been depleted before t	0%	6.6% = 1/15
Ruin probability, with stopping when the equity has been depleted before t	0%	20% = 3/15

These models grow out of the so-called ruin theory, which posits mathematical models designed to describe an insurer’s vulnerability to insolvency or ruin. Here they are explained on the basis of solutions to differential equations like the heat equation. In this approach, the stopping hypothesis will be regarded as fulfilled by the solutions of the heat equation vanishing on the equity-zero axis, which means the probability that the path taken by the development of the equity capital has ever crossed the zero line in the past is zero. This is a geometrically more tangible approach than the usual derivation found in the actuarial literature,

cf. e.g. Søren and Albrecher (2010), which is little known outside the actuarial community, probably because it is very technical and not intuitive.

3. Numerical Results

a. A Sketch of the Equity Process Model

In order to emphasize the importance of the assumptions regarding the time horizon and the hypothesis of stopping, we start by presenting numerical values for the probability of ruin along with the probability of a shortfall for some examples of stochastic processes describing the equity of an insurance company. These stochastic processes are defined as Brownian motions with an initial equity ratio d , return margin m and volatility σ . If we were to consider the ruin probability, the stochastic process would stop when the equity was depleted before the time horizon. However, there is no stoppage of the Brownian motion of the equity when we calculate the shortfall probability. This means the random walks of the equity with negative values before the time horizon are not allocated to any shortfall if the equity is positive at the end (i.e., at the considered time horizon). In this model, the probability of being in shortfall can be calculated by simply evaluating a normal distribution with a standard deviation $\sigma\sqrt{t}$ at minus the expected value of the equity at time horizon t , that is, at $-d(+mt)$. Therefore, with the cumulative normal distribution denoted by Φ , the shortfall probability is

$$\Phi\left(-\frac{d+mt}{\sigma\sqrt{t}}\right).$$

For the calculation of the ruin probability, an additional probability must be taken into account that covers the cases ignored by the default probability—i.e., the cases in which the stochastic equity process was stopped at some point in the meantime but then recovered and shows positive equity up to the assumed time horizon. This additional probability can also be expressed by a cumulative normal distribution, in fact by

$$e^{-\frac{2md}{\sigma^2}} \Phi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right).$$

This term becomes increasingly important with a growing time horizon. This is reasonable because the probability of previously insufficient coverage increases, whereas the probability of a shortfall without a prior stop decreases after a certain time. While the probability of a shortfall at a point in time t without stoppage before can already be understood by fairly straightforward calculations, the derivation of the latter term, which takes into account a stoppage due to the prior

undercutting of the zero-line, usually requires very advanced probability theory—hardly accessible to non-specialists.

b. An Illustrative Example of the Equity Walks in the Two Different Concepts

Figure 1 on page 5 gives an illustrative example to visualize the two different models. The random walk of the equity starts at time 0 at a given positive initial equity $d > 0$. We expect the equity to drift upwards due to a positive trend parameter, the return margin $m > 0$.

c. Numerical Values and Their Discussion

Table 1 considers the combinations of two different values for the initial equity and the return margin and the same volatility everywhere. In the probabilistic model of a Brownian motion, this indicates:

- The probability of shortfall without stopping when the process reaches the zero line.
- The probability of ruin with finite and infinite time horizon with stoppage when the equity was previously depleted.

Table 1

Stoppage when the equity has been depleted before t :					Probability of shortfall "Simple model"				Probability of ruin $\psi(t)$ "Advanced model"			
					without				with			
Parameter set Ps	Initial equity d	Return margin m	Volatility σ	Formula	$\Phi\left(-\frac{d+mt}{\sigma\sqrt{t}}\right)$				$\Phi\left(-\frac{d+mt}{\sigma\sqrt{t}}\right) + e^{-\frac{2md}{\sigma^2 t}} \Phi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right)$			
				time horizon = years t	1(*)	5	20	100	1	5	20	∞
1	10%	1%	4%		0.30%	4.68%	4.68%	0.30%	0.65%	12.9%	25.1%	28.7%
2	20%	1%	4%		0.00%	0.26%	1.27%	0.13%	0.00%	0.6%	5.4%	8.2%
3	10%	2%	4%		0.13%	1.27%	0.26%	0.00%	0.32%	5.4%	8.1%	8.2%
4	20%	2%	4%		0.00%	0.04%	0.04%	0.00%	0.00%	0.1%	0.6%	0.7%

The standard formula for the Solvency II requirement is based on the specifications (*). They simply represent the probability of shortfall at the end of the one-year period for an equity following the stochastic process of a Brownian motion. These values show:

- i. All the four examples meet the Solvency II requirement with a probability of shortfall of less than 0.5%—i.e., a level of solvency that can withstand a one in 200-years event.
- ii. The values of the probability of ruin with a longer time horizon are much higher than those of Solvency II. This indicates that the seemingly very strong Solvency II requirements with a probability of shortfall of at most 0.5% are largely due to the specific assumption upon which the model is based—namely, the one-year time horizon.
- iii. For a broad time horizon of 20 or 100 years, concepts based on the shortfall of a normally distributed equity at the end of these periods are meaningless. By then the riskiest time for the company could be over. All that can be checked is whether the company is solvent exactly at this point in time—e.g., after 20 years—regardless of what happened before. Let us take the example of parameter set 3 in Table 1: In the model that considers the shortfall, there is no stopping when the equity is depleted before. Then only 0.26% of the companies are insolvent after 20 years, but $8.1\% - 0.26\% \approx 7.8\%$ have become insolvent at least once in this period of 20 years and have then recovered to report positive equity again at the end of the 20 years. Thus, when considering longer time horizons, the simple shortfall model is no longer applicable, and the advanced model needs to be applied.
- iv. The parameter set 2 assumes an initial equity, which is twice as large as the one of parameter set 3 but has only half of the return margin. These effects on the probability of ruin cancel out with infinite time horizon. However, in order to meet the Solvency II requirements, the initial equity is much more important than the return margin. This is mainly due to the small time horizon for Solvency II, which means the return margin can only contribute to equity growth for one year.

Thus, it is important to note that the too narrow time horizon of Solvency II may lead to a too risk-averse business strategy, underestimating the importance of a reasonable margin in the longer term for life insurances. Another way to explain this is as follows: The risk grows with the square root of time, and the additional return due to a good margin grows linearly over time. For example, if you compare a period of 20 years with a period of one year, the risk increases only about 4.5 times, whereas the additional yield increases 20 times. But the reality is a little more complicated: If you do not have enough equity at the beginning, you starve on the way, and it is precisely this risk that is also taken into account in our advanced model.

4. The Calculation of the Shortfall Probability Using the Standard Normal Distributions

Here we consider a stochastic process, defined as Brownian motion, beginning with an initial equity ratio d constantly growing with a positive return margin m and carrying a risk corresponding to a volatility σ . With these assumptions, the stochastic equity process will not be stopped when the equity is depleted—i.e., when the equity crosses the zero line and slides into the negative range due to an unfavorable risk development. Brownian motions are the most common stochastic processes. They can be interpreted as diffusion processes, which describe the spreading of heat as time passes. Those processes are well-known in physics, especially in thermodynamics. The probability distribution of a random variable defined by Brownian motion fulfills a partial differential equation (PDE) of a type referred to as heat diffusion equation. See equation (9) and (10) in chapter 7 below. In our case, the random variable corresponds to the equity of the considered insurance company, which starts at time 0 at the given initial equity d and then spreads more and more as time passes due to the volatility σ analogue of the diffusion rate or the thermal diffusivity in the physical context.

The function

$$p_+ = p_+(x, t) = \varphi_{\mu, \sigma\sqrt{t}}(x) = \varphi_{d+mt, \sigma\sqrt{t}}(x) = \eta \cdot e^{-\frac{1}{2} \frac{(x-d-mt)^2}{\sigma^2 t}},$$

$$\eta = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{t}} \tag{1}$$

solves the heat equation of type (10) and defines the probability density of the equity at time t . Thus, the probability of a negative equity at time t corresponds to that indicated above in the simple model, which only considers a shortfall at a given time t , without stoppage due to the premature depletion of equity:

$$\int_{-\infty}^0 p_+(x, t) dx =$$

$$\int_{-\infty}^0 \varphi_{d+mt, \sigma\sqrt{t}}(x) dx = \int_{-\infty}^{-(d+mt)} \varphi_{0, \sigma\sqrt{t}}(x) dx = \int_{-\infty}^{-\frac{d+mt}{\sigma\sqrt{t}}} \varphi_{0,1}(x) dx$$

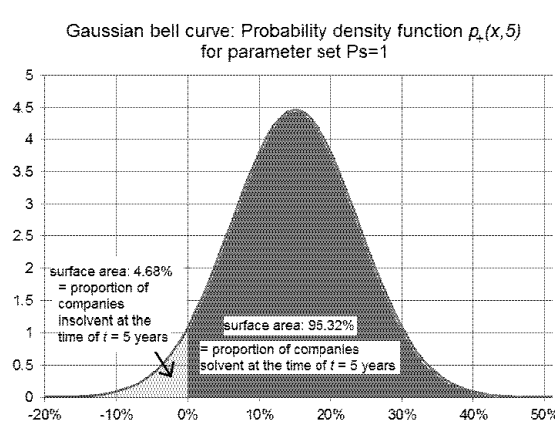
$$= \Phi\left(-\frac{d+mt}{\sigma\sqrt{t}}\right). \tag{2}$$

5. The Additional Probability for Previously Stopped Equity Processes

a. Some Remarks About the Gaussian Bell Curve

For a given time $t > 0$, $p_+(x, t)$ has the shape of a bell. Indeed, this is the famous Gaussian bell curve as shown in Figure 2.

Figure 2



Parameter set Ps 1:

Initial equity $d = 10\%$
Return margin $m = 1\%$
Volatility $\sigma = 4\%$

$p_+(x, 5)$ shows the probability density of the equity of a firm, whose equity moves with parameter set 1, at the time after five years, without going out of business when the equity was depleted before during this five years. In this case as listed in Table 1, the probability of underfunding is 4.68%.

This probability can be visualized by the area of the surface between the probability curve and the x-axis for the negative equity range.

b. The Additional Probability for the Stopped Equity Processes

Gaussian bell curves run from $-\infty$ to $+\infty$. Thus, the equity can attain any negative range with positive probability. This does not correspond to the usual public requirements for the available equity, where a negative equity is strictly inconceivable. An insurance company goes out of business immediately when its equity moves down to zero. We are looking for another solution of the diffusion equation that we have not considered yet. For this new solution, we impose an additional boundary condition on the equity-zero axis—i.e., on the line $(x, t) = (0, t)$, $t > 0$. As is shown in the Appendix 1, the function

$$p_-(x, t) = e^{-\frac{2dm}{\sigma^2}} \varphi_{-d+mt, \sigma\sqrt{t}}(x) \quad (3)$$

also fulfills the PDE (10) in chapter 7, and it pushes moreover the Gaussian bell curves p_+ down, so that the difference $p = p_+ - p_-$,

$$p = p(x, t) = p_+(x, t) - p_-(x, t) = \varphi_{d+mt, \sigma\sqrt{t}}(x) - e^{-\frac{2dm}{\sigma^2}} \varphi_{-d+mt, \sigma\sqrt{t}}(x) \tag{4}$$

fulfills the boundary condition, the vanishing of $p(x, t)$ on the $x=0$ -axis. This makes sense, because p only considers the firms whose equity had never become negative. The subtrahend p_- can be interpreted as the probability density for an equity $x > 0$ at time $t > 0$, whereby the equity was exhausted at least once in between, but then recovered again. In other words, the probability density that the equity process attains the equity $x > 0$ at time $t > 0$ but has crossed the “red” line of ruin $x = 0$ once before. To determine the proportion of companies that have left the business because they previously reached the “red” line and lost all their equity, one has to calculate how much the area of the Gaussian bell curve p_+ has been reduced by the subtrahend p_- . This reduction in area has to be restricted to positive equities > 0 , thus to the companies still in business, which yields the following integral to determine the additional probability:

$$\begin{aligned} \int_0^\infty p_-(x, t) dx &= \int_0^\infty e^{-\frac{2dm}{\sigma^2}} \varphi_{-d+mt, \sigma\sqrt{t}}(x) dx = e^{-\frac{2dm}{\sigma^2}} \int_0^\infty \varphi_{-d+mt, \sigma\sqrt{t}}(x) dx \\ &= e^{-\frac{2dm}{\sigma^2}} \int_{\frac{d-mt}{\sigma\sqrt{t}}}^\infty \varphi_{0,1}(x) dx = e^{-\frac{2md}{\sigma^2}} \Phi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right) \end{aligned} \tag{5}$$

To extend the simple model of shortfall probabilities, which ignores previous underfunding to the advanced model of ruin theory, this term has to be added. As mentioned above, especially for longer time horizons, this additional probability becomes increasingly important and must be taken into consideration.

The probability of ruin up to time t , noted by $\psi(t)$, results as the sum of the probability of the shortfall and the additional probability, thus

$$\psi(t) = \Phi\left(-\frac{d+mt}{\sigma\sqrt{t}}\right) + e^{-\frac{2md}{\sigma^2}} \Phi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right). \tag{6}$$

The first term of the right side of equation (6) corresponds to the probability of shortfall. See equation (2)—i.e., the probability of having negative equity at time t , irrespective of the development of equity in the period from 0 to t . The second term represents the additional probability, cf. equation (5)—i.e., the probability that the entire equity capital has been lost at least once in the meantime, but has then recovered to positive equity if the business could have continued. Looking at the illustrative example in Figure 1 for $t = 10$ years, the first term corresponds to the probability of $1/15$ and the second term to $2/15$, which add up to the ruin probability $3/15 = 1/5$.

For an infinite time horizon, the ruin probability $\psi(\infty)$ corresponds to the limit of $\psi(t)$ when t approaches infinity. Because the first term then describes the cumulative normal distribution at minus infinity and the second one at plus infinity, the first term vanishes and the second term gives the factor, which

multiplies 1, the cumulative normal distribution at plus infinity. Hence, the ruin probability results to ⁵

$$\psi(\infty) = e^{-\frac{2md}{\sigma^2}} \quad (7)$$

The probability of staying in business forever, also referred to as survival probability, is

$$1 - \psi(\infty) = 1 - e^{-\frac{2dm}{\sigma^2}} \quad (8)$$

and the probability of staying in business up to time t —i.e., the survival probability with finite time horizon t —is $1 - \psi(t)$.

6. Graphics

a. Geometric Interpretations of Shortfall and Ruin Probabilities by the Area of Surfaces

The most important and applied parts of mathematics often combine two very different concepts, providing a synthesis of some vague geometric idea with a strict and unambiguous formalism. Just think of differential calculus and the interpretation of the integrals as the surface area. Hence, the probabilities of shortfall in the simple model of a Brownian motion, as provided for by Solvency II, are usually explained by the area of the surface between a Gaussian bell curve and the x-axis up to the considered Value at Risk. This gives a similar picture as in Figure 2, but with a much smaller area to represent insolvent companies for the Solvency II requirements of only 0.5%.

5. Similar formulas, at least for the upper limits of the probability of ruin, also apply to more general risk processes. This allows, for example, jumps in order to model particularly high individual claims as they may typically occur for non-life coverages without appropriate reinsurance protection. In this context, the parameter $R = 2m/\sigma^2$ usually denotes the “adjustment coefficient” because of the way in which it can be calculated for these more general risk processes. In the more general context, it also depends on the return margin and on the appropriate parameters describing the supposed risks of the process. These classical considerations are mainly due to two Swedish actuaries and statisticians: Filip Lundberg introduced his theory at the beginning of the 19th century, and Harald Cramér republished part of Lundberg’s work in the 1930s. Before the introduction of Solvency II and the Swiss Solvency Test (SST), the theory of ruin seemed to be more relevant for risk management. In this context of risk management, ruin theory applies to the decisions regarding how much risk one is willing to take and which part should be reinsured with a given equity ratio. As already mentioned, it is regrettable that this broader view—as opposed to the narrow view based on the very short one-year time horizon required by the standard formula in Solvency II and in the SST—seems to have lost attention when implementing the complicated and meticulous new European regulations.

We now seek such interpretations for the ruin probabilities, thus attempting to interpret the surfaces between the curve p and the x-axis as probabilities of ruin. Figure 3 illustrates the distribution of the equity $p(x, t)$ depicted as curves for fixed time horizon.

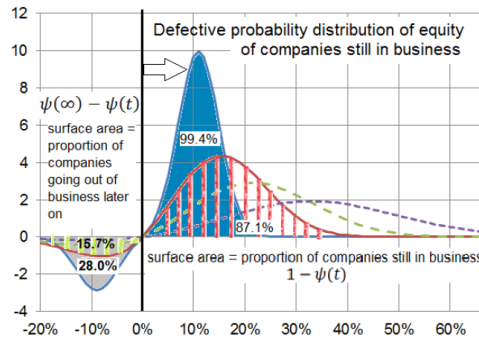
Figure 3

Parameter set: **Ps 1**

Initial equity $d = 10\%$
Return margin $m = 1\%$

In the first year, 0.6% of the companies were ruined.

Between the first and the 5th year
 $99.4\% - 87.1\% = 28\% - 15.7\% = 12.3\%$
were ruined.

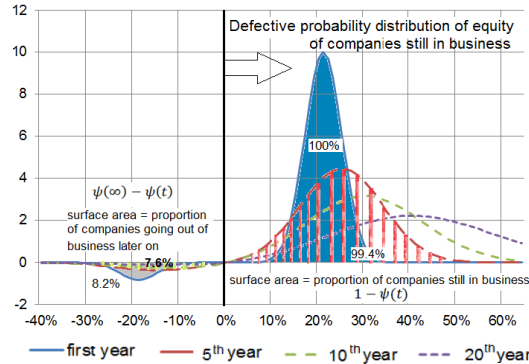


Parameter set: **Ps 2**

Initial equity $d = 20\%$
Return margin $m = 1\%$

In the first year, 0.0%⁶ of the companies were ruined.

Between the first and the 5th year
 $100.0\% - 99.4\% = 8.2\% - 7.6\% = 0.6\%$
were ruined.



Types of lines

— first year — 5th year — 10th year — 20th year

For a given $t > 0$, the curve $p = p(x, t)$ on the positive side, that is for positive x , describes the probability density of the equity for the companies that are still active. Strictly speaking, this function p on the positive side does not represent a probability distribution function, because the sum of the probabilities does not add up to 100%. This is because some companies prematurely left the business. In Feller’s classic textbook (1971), the probability distributions that add up to less than 100% are referred to as “defective.”

The curve $p = p(x, t)$ on the negative side has no immediate interpretation. Indeed, insurance companies are not allowed to have negative equity. As soon as equity reaches zero, the process will be stopped, which means that the company must cease its activity. But there is another interpretation of the negative side of the curve p : This part of the curve does not provide information about the past, but about the future development of equity, in particular on the probability of ceasing activity in the future ($\tilde{t} > t$). The surface area on the negative side between the

curve p and x-axis corresponds to the probability of going out of business later on, thus for $\tilde{t} > t$, because the surface area may be calculated by an appropriate integral that gives $\psi(\infty) - \psi(t)$.

Figure 3 represents the probability distribution of the equity $p(x, t)$ for $t = 1, 5, 10$ and 20 years and parameter set 1 and set 2 on the positive side of the x-axis. As already mentioned, the curves on the positive side describe the probability density of equity for companies that are still active. The longer the time horizon under consideration, the more the equity spreads and the curves become flatter and flatter. The area of the surface between the probability density curve and the x-axis on their positive side decreases as the considered time interval becomes longer, since more companies have been at least once underfinanced over a longer time period and had to leave business. The difference between the two areas on the positive and negative sides of the x-axis is constant over time, as can be seen in Table 2. This difference, also referred to as survival probability, shows the proportion of companies that remain solvent for any length of time from the start at time 0—i.e., the survival probability $1 - \psi(\infty)$.

Table 2 lists further values for the area contents of the curves in graphic 3 that were not entered there.

Table 2

t years	Parameter set: Ps 1, d = 10%, m = 1%			Parameter set: Ps 2, d = 20%, m = 1%		
	Proportions of companies		Survival probability	Proportions of companies		Survival probability
	still in business after t years = $1 - \psi(t)$	going out of business sometime after more than t years = $\psi(\infty) - \psi(t)$		still in business after t years = $1 - \psi(t)$	going out of business sometime after more than t years = $\psi(\infty) - \psi(t)$	
(I)	(II)	(I)–(II)	(I)	(II)	(I)–(II)	
1	99.35%	28.00%	71.35%	100.0%	8.2%	91.8%
5	87.07%	15.72%	71.35%	99.4%	7.6%	91.8%
10	79.98%	8.63%	71.35%	97.4%	5.6%	91.8%
20	74.93%	3.58%	71.35%	94.6%	2.8%	91.8%

The parameters for both sketches correspond to the same return margin $m = 1\%$ p.a. and to the same volatility $\sigma = 4\%$ p.a., but at different initial equities. Therefore, the areas on the negative side are considerably higher for Ps 1. Because the curve p for Ps 2 intersects the x-axis at point 0 in a very shallow angle, it will take some time until a major part of the ruin cases occurs. In fact, in the case Ps 2, a time horizon of one or even five years does not give any good indication of the complete probability of ruin with infinite time horizon. In this case, the fairly high initial equity combined with a small margin leads to a high value for the expected time of ruin μ_{IG} , assuming that ruin ever occurs. See also, for example, Table 3 and Figure 4.

Figure 5 in Appendix 3 on page 28 displays the complete functions $p(x, t)$ as a two-dimensional surface in three-dimensional space and not just vertical sections of this surface for some selected time points.

b. The Physical Interpretation of Figure 2 and Figure 3

Looking at these Brownian movements under physical circumstances, it can be noted that the area between the curve describing the heat distribution over an interval of the x-axis corresponds to the heat stored in this sector of the x-axis for a given point in time. If you look at the entire x-axis from minus infinity to plus infinity, the heat cannot escape anywhere and thus remains constant over time. With the arithmetic Brownian movement, this area always remains equal to the total probability of 1, as shown in Figure 2.

The situation in Figure 3 is somewhat more complicated, but also has a nice physical interpretation through a heat pool on the positive x-axis and a cooling pool on the negative x-axis. Over time, the cold pool moves to the right and mixes with the warm pool. The total heat in the system always remains the same—i.e., the area difference above and below the x-axis is constant over time and corresponds to the constant survival probability $1 - \psi(\infty)$ in our solvency model according to Table 2. This physical interpretation is especially vivid in the case without any margin: Then both the areas above and below the x-axis are even equal, or in the physical picture, the heat and cold accumulators are equal. As a result, all heat is lost over time, which in our economic context means that the probability of the company remaining solvent is zero, and the probability $\psi(\infty)$ of going into ruin at some time is one.

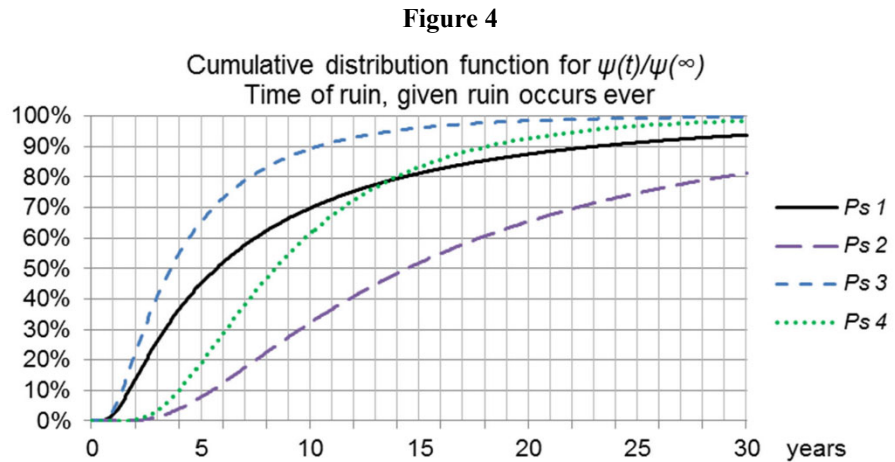
c. The Distribution of the Time of Ruin Given That Ruin Ever Occurs Shown in Figure 4

The formula for the expected time of ruin μ_{IG} can be obtained by $\psi(t)/\psi(\infty)$ as an Inverse Gaussian distribution $IG_{\mu_{IG},\lambda}$ with its two parameters μ_{IG} and λ , which leads to the simple relation $\mu_{IG} = d/m$. (See Appendix 2 for more details.) For the case of Ps 2, it shows that it takes more time to get out of the danger zone of a possible ruin. In the case of Ps 1 or even Ps 3, the dangerous time runs out faster, but here too it lasts more than a year. The particular values of μ_{IG} for the parameter sets considered are listed in Table 3:

Table 3

Parameter set Ps	Initial equity d	Return margin m	Expected time of ruin, assuming it occurs ever, in years, $\mu_{IG} = d/m$
1	10%	1%	10
2	20%	1%	20
3	10%	2%	5
4	20%	2%	10

Figure 4 shows the cumulative probability distribution function $\psi(t)/\psi(\infty)$ of the time of ruin for the four parameter sets.



d. The Use of the Inverse Gaussian Distribution in the Theory of Stopped Brownian Motions

In modern textbooks, the probability of finite-time ruin for a Brownian motion with drift is usually derived by transforming the process into a new process corresponding to Wald's martingale and then applying the stop theory to this transformed process in order to obtain the moment-generating function of the probability distribution of ruin, given that it occurs ever. This moment-generating function is then transformed by the inverse Laplace transformation, and this, after all those calculations, leads to the Inverse Gaussian distribution function in the form set out at the end of Appendix 2. This requires a great deal of hard work and a deep understanding of advanced mathematics, cf. for instance Crépey (2013). Of course, this provides only a formal proof, without any intuitive and more quantitative understanding of what actually happens to the Brownian process while stopping when it crosses the given "red" line of ruin. It is, therefore, not surprising that the laws and guidelines of Switzerland and the European community and the planned global capital standard of the IAIS are not based on these concepts of the theory of ruin, but on much simpler approaches such as those of Solvency II. The latter ones, based on a value-at-risk ratio for a time horizon of one year, also seem to be comprehensible to non-specialists, a necessity for legally founded solvency regulation.

7. More About the Differential Equation Modeled on the Heat Equation

In general, the term “diffusion” describes a dynamic process whereby physical particles spread out from an area of high concentration to an area of low concentration. These particles can be atoms or molecules or, in an economic context, assets like the price of shares or the equity of a company. In the economic context, the probability of a given asset spreads out from a known current value to distributions of potential subsequent values. These distributions are very steep at the beginning of the process. Then they melt away and spread out over a wider area with time. (See Figure 5 and Figure 6 in Appendix 3 on pages 2829.) In this context, the diffusion process acts on the probability distribution of economic values, showing the uncertainty of future developments by spreading out from known fixed values to more and more different potential values. The faster the spreading takes place, the riskier the equity is considered. Usually, the economic parameter describing this risk of spreading out is denoted as “volatility,” as we do in this paper.

In physics, the most notable diffusion process describes the evolution of the repartition of heat in a medium. The heat will be conducted from “hot spots” toward colder regions. The mathematical description is given by the heat conduction equation, a partial differential equation describing how the repartition of heat develops as time passes by. The more concentrated the temperature⁶ u , the faster the heat will be conducted away, measured by a decline of $-\partial u/\partial t$ in the temperature. This heat-concentration is gauged by the second derivative of the repartition of temperature in the medium preceded by a negative sign, $-\partial^2 u/\partial x^2$. Therefore, $\partial u/\partial t$ and $\partial^2 u/\partial x^2$ are proportional, linked by a positive constant of proportionality. In the case of the heat equation, this constant depends on the material of the medium considered. Since we are not interested here in actual heat equations and their solutions, we move on to the economic context and understand this constant as volatility σ of the equity considered. In order to emphasize that the diffusion process refers to probability distributions, we also drop the notation of u commonly used to describe the heat equation and use the letter p instead, suggesting that we are dealing with probability distributions. In this paper, the PDE of this type of heat equation,⁷

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x^2}, \tag{9}$$

6. The commonly used letter u in thermodynamics denotes the *internal energy*, a more general term than the temperature. In the considered simpler setting, u is proportional to the temperature and may be understood as such.

7. The constant of proportionality is selected $\sigma^2/2$ to ensure that the notation fits with Brownian motions with volatility σ .

will be considered as a diffusion equation. Equation (9) describes how the uncertainty of the future development affects the value of an asset and transforms this value to an entire range of possible values to be taken into account when dealing with risk management.

Since we are particularly interested in the economic risks over a long period of time, we have to take into account an additional component reflecting a continuous increase in the value of the asset due to an investment return—i.e., a margin m in the business granted by the equity in question. This leads to the partial differential equation

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x^2} + m \frac{\partial p}{\partial x}, \quad (10)$$

describing a diffusion process analogous to the development of the probability density of a Brownian motion with volatility σ , which can be understood as “risk intensity” (risk exposure per time⁸) and by a drift m caused by a return margin rate m . This rate designates the expected rate of return margin per time t . In the usual economic context, the standard measurement unit for the time t is years, and thus the two development parameters σ and m are given in change per year.

Since we are looking here at the development of equity and not, for example, the development of assets or liabilities separately, we do not consider here geometric Brownian motions that lead to lognormally distributed random variables when viewed at a fixed point in time.

8. The Two Risk Measures: Value at Risk (VaR) and Conditional Tail Expectation (CTE)

This topic is of secondary importance for the considerations in this article and can be solved simply by recalibration. However, it is explained here for a better overall understanding.

The issue is that one of the two risk measures used in connection with the solvency requirements must be selected. One of these, the so-called VaR, requires compliance with a level of trust within which the insurance company may not become insolvent. This method was adopted by Solvency II and requires the insurance company to remain solvent in a 99.5% percentile over a one-year period. The required capital at risk with this specified percentile is called VaR, and it

8. The variance σ^2 has the unit “currency squared per time,” which in the probabilistic interpretation corresponds to an increase of the variance of the considered random equity per time. In financial sciences, the volatility σ is usually given based on a currency unit “1” and, as mentioned below, the time coordinate in years.

corresponds to the capital buffer required to remain solvent within the desired confidence interval.

The second method is called expected shortfall or CTE. As with the VaR method, it is necessary to remain solvent with a certain percentile, but it also requires sufficient funds to absorb the average load in the unfavorable cases outside this percentile. This risk measure is applied to variable annuities by the U.S. RBC standard and requires a risk charge for the stochastically calculated market risk C-3c based on a CTE of 90%.

In this context, it is particularly interesting that this is taken into account not only for a time horizon of one year, as is the case for almost all stochastic solvency requirements, but in accordance with the recommendations of the American Academy of Actuaries (Academy) (2005) over a period of decades until most policies have expired. Since fat tail risks are generally not reflected for market risks, the VaR and CTE risk measures are linked by appropriate recalibration. According to the Academy, the CTE 90% risk buffer required for variable annuities corresponds to approximately a VaR 95% demand. Thus, the C-3c component of the U.S. RBC requires that variable annuities have enough risk capital to ensure that the probability of insolvency is above 95%. Conversely, the probability of insolvency is below about 5% and this does not refer, *nota bene*, to the time horizon of one year, but until the policies expire. There is extensive literature dealing with various methods for the calculation of the risk measures for variable annuities. We will discuss this literature in more detail in Section 9.c. and compare it with our model.

9. Comparison of Our Model with Those in Financial Literature

a. Comparison with Similar Stochastic Models

Myers and Read (M&R) (2001) also reported numerical values for the case where the stochastic component was not only modelled by two separate lognormal distributions for assets and liabilities, as this is the case for a series of similar studies mentioned in the introduction, but additionally also by a normally distributed equity. This corresponds to our approach with regard to the stochastic component, if you only look at the shortfall probability time $t = 1$ with a vanishing return margin $m = 0$. Their studies, as well as the similar ones, are especially interested in the default value, also referred to as default ratio. This default value corresponds to the additional requirement in the case of a CTE metric instead of a VaR metric. However, in the case of solvency requirements, each insurance undertaking must provide this capital cushion for its own hypothetical insolvency, while this default value discussed in the financial literature is understood as general costs for the insurance industry due to such insolvencies. The additional

equity requirement for the CTE metric corresponds, therefore, to the default value as referred in the financial literature divided by the probability of the considered shortfall. As a reminder, Solvency II applies a VaR metric, while the Swiss Solvency Test (SST) and the U.S. RBC for variable annuities are based on a CTE metric. To make these relationships between our model and the similar model in M&R clear, we present here some figures of Table 5 of the M&R study for a normally distributed equity using our notations.

Table 4

		M&R	Solvency II	SST
Equity requirement for VaR metric	d	50%	75%	68%
Volatility (referred to liabilities)	σ	29.18%	29.18%	29.18%
Probability of shortfall	$\Phi(-d/\sigma)$	4.331%	0.500%	1.000%
Default value	$-d \cdot \Phi(-d/\sigma) + \sigma \Phi'(d/\sigma)$	0.52%	0.05%	0.10%
Additional equity requirement for CTE metric	$ad = \text{Default value} / \text{probability of shortfall}$	12%	9%	10%
Entire equity requirement for CTE metric	$d+ad$	62%	84%	78%

All columns in Table 4 are based on the volatility listed in Table 5 of the M&R paper. The column referred to by M&R also assumes equity capital, called surplus in the M&R paper, of 50% of the liabilities, which corresponds to $d = 50\%$ in our notation and leads to a default value of 0.52% as listed in the M&R study. This corresponds to the expected value of defaults, i.e., the expected value of a negative equity at the time point of one year. The capital cushion for solvency requirements with a CTE metric also includes this expected value of default in the hypothetical case of a shortfall. In the M&R column, this results in $0.52\%/4.331\% = 12\%$ additionally required equity for the stricter CTE metric. However, the capital requirements in the M&R column are below those of Solvency II or the Swiss SST. With this volatility of 29.31%, as assumed by M&R, Solvency II would require the equity to be increased to 75% of the liabilities such that the capital cushion meets the demanded 0.5% confidence level for a VaR metric. Accordingly, the Swiss SST would require equity capital of 78% of the liabilities so that the requirements of a CTE metric are met with a confidence level of 1%, cf. FINMA (2006).

b. Differences Between Our Economic Parameters and Those in the Financial Literature

We assume a significantly lower volatility of equity than in the above-mentioned M&R study, as well as in the papers referred to in the introduction, cf. Stoyanova and Schlütter (2015) and Yow and Sherris (2008), where volatilities of

20% and above are assumed. We believe that these economic parameters do not or no more correspond to reality, at least not to those shown by the reports for Solvency II.

According to the balance sheet and the equity published in EIOPA (2017) for the third quarter of 2017 in regard to the entire EU market, equity amounted to around 15% of liabilities or 13% of assets and the solvency ratio to around 240%. In simplified calculations, this results in an average solvency capital requirement (SCR) of around $15\%/240\% = 6.25\%$ of liabilities; the actual value corresponds to 6.5% of liabilities. The confidence level for Solvency II of 99.5% corresponds for a normal distribution to the value at risk of 2.57 times the standard deviation. Thus, the average volatility of equity for the overall market under Solvency II amounts to about $6.5\%/2.57 \approx 2.5\%$ of liabilities, which is even below our assumption of 4% and far from the level of more than 20% often used in financial literature.

In this context, however, it should be noted that the financial literature quoted here refers to typical non-life insurance companies, while the EIOPA Solvency II figures refer to the market as a whole, where the generally larger balance sheet volume of life insurers is decisive for the overall figures. In the non-life insurance industry, the insurance risk itself is much more decisive than the investment risk, which leads to a significantly higher overall risk for the non-life insurances when the standard deviation measuring the risk is expressed in relation to the assets volume. As already mentioned above, there are significant differences between the characteristics of the risks assumed by a life insurer and those borne by a non-life insurer, such as the completely different contract durations.

c. Comparison with Calculation of Risk Measures for Variable Annuities in the Literature

We pointed out in Section 2 that the U.S. RBC standard for variable annuities takes into account the entire period up to the expiry of the contracts, which we consider exemplary, and mentioned in Section 8 that there is extensive literature on the calculation of risk measures for variable annuities. In the following, the concepts, methods and results for the variable annuities discussed in the literature are compared with those of our model.

Conceptually, there is a difference between our model and that of many of the papers published in the literature: Both our model and the U.S. RBC requirements take into account the risk that policyholders will receive insured benefits from the insurance company. In contrast, the literature usually calculates the risk taken by the insurance company on specific insurance contracts, which provide nominal guarantees for investment vehicles that are close cousins of investment funds, cf. Milevsky and Salisbury (2006). This paper, like others—such as Feng (2014), Feng and Volkmer (2012), and Bauer, Kling and Russ (2008)—deal mostly with a fixed maturity and evaluate the probability distribution of the volatile investment part less the nominal guarantees. They usually only consider the outcome of the

stochastic process at maturity—i.e., at the end of the contract period—and thus neglect the course of the random path during this contract period. In particular, if the process is very long, the risk almost disappears if you look only at the end point and if your guarantee is based on lower interest rates than the expected returns of the investments, what is fulfilled by a reasonably designed insurance product. The significant apparent risk reduction by considering the process only at the end of a long period is shown in Table 1 of Section 3 by the probability of a shortfall for long periods of 20 or 100 years.

From the insurer's point of view, it is right to ignore the stochastic process in the meantime, if the calculation takes correctly into account all guarantees that the policyholder holds during the term of the contract. The situation is quite different if you take a regulatory perspective. In this regard, the principle of hope, saying here that with good luck the insurance will come out of a deficit sometime later, is out of place. If the financial situation of an insurance company becomes insufficient at some time, it must leave business immediately so that new customers can always be sure to conclude contracts only with financially healthy companies. This makes the stopping hypothesis crucial, especially if longer periods are taken into account for solvency requirements, as we believe to be correct, in particular for life insurance policies, which generally have long maturities.

In the case of variable annuities with guaranteed minimum death benefits (GMDB), the two perspectives almost give the same picture. See Gerber, Shiu, Yang (2012). When death occurs, the guaranteed benefit must be paid immediately, and the insurance company cannot wait until the contract expires. With this coverage, the stochastic investment process is stopped during the contract term and not at the end of the period, regardless of any regulatory requirements. This is similar to the stopping hypothesis in our model. Therefore, it is not surprising that the prices of GMDB increase when longer periods are considered. See Table 2 and Table 3 in the paper mentioned, similar to the figures in our Table 1 for the probability of ruin.

Apart from this methodological difference due to the stopping hypothesis applied here, there are also important similarities to all the papers mentioned here; they also take into account a much longer time horizon, usually between five and 20 years to the maturity of the contracts in question, instead of just one year as with Solvency II. Of course, these longer time intervals then lead to significantly lower security levels of 80%, 90% or 95%—i.e., far below the high level of 99.5% of Solvency II. As our Table 1 shows, the time horizon is decisive for any risk assessment, and the high level of 99.5% appears unrealistic when calculating the risk for the entire term for these variable annuities guarantees. The studies show in particular that the risk decreases significantly as the margin increases. See for instance Table 3 in Milevsky and Salisbury (2006).

In terms of one technical, mathematical aspect, all these papers, including the presented one, are similar. Namely, at a particular point in time, you know exactly what the facts are, which then become more and more uncertain the further you move away from that particular point. In our model, you know exactly the equity

capital at the beginning of the process at time 0. In the mentioned financial literature, this particular point is at the end of the period, thus at the maturity time T , when the warranted guarantee no longer represents a risk, but has become a fact, a loss, if the value of the investment is not sufficient to cover the guarantee or no loss otherwise. This unambiguous situation at time 0 or at time T is then viewed forward at time T in our model or backwards at time 0 in the cited literature in order to calculate the current price or risk of a guarantee warranted by a variable annuity product. As mentioned, the clear situation becomes more and more diffuse when you move away from these particular points in time. This diffusion is modelled by a diffusion process and mathematically described by a PDE similar to the heat equation here. In using today's computer technology and new and sophisticated numerical methods to solve these PDEs, one may speed up the often time-consuming calculations, cf. Feng (2014) and Privault and Wei (2018). Note that all these calculations can be done with Monte Carlo simulations, which in practice are still the most common approach. The advantages of the Monte Carlo method are that it can be adapted almost arbitrarily to the specific assumptions regarding product design and policyholder behavior, and it can be used without deep mathematical knowledge. In contrast, the disadvantage is that they are time-consuming without conveying as clear an insight into the crucial assumptions and parameters as can be gained by analytical solutions.

10. Conclusion

Our calculations and analyses show that in addition to volatility and initial capital, the business margin is becoming increasingly crucial when considering solvency over a longer time horizon. It is unclear whether the global trend in setting solvency requirements for insurance companies takes this into account or follows the path taken by the EU with Solvency II, see IAIS (2017), which will be particularly problematic for the life insurance industry.

For the management of life insurance policies, for which a time horizon of several decades is the rule, the short time horizon of the standard formula within the framework of EU Solvency II can entail an investment policy that is too risk-averse. In the broader range, it may lead to a too cautious business management at the expense of higher business margins, which gain in importance only when looking at longer time horizons. It is also questionable to encourage insurance companies to issue hybrid capital in order to reduce the risk to the detriment of the margin. Since margins mainly count in the long run, regulations requiring the use of such instruments could in fact only reduce the apparent risks measured by mandatory risk metrics without actually helping customers with a long-term perspective.

The high confidence level of 99.5%, which Solvency II exhibits, results from the way risks are measured there, and this relates, in particular, to the short time horizon of only one year. For those life insurance customers who are not aware

that this high level corresponds only to a one-year perspective, this may result in a too favorable picture of the confidence level that is of interest to them—i.e., for the entire contract term of possibly several decades.

Our proposed model for regulating the life insurance industry takes into account the entire contract term and focuses not only on the end of the contract, when a sufficiently high margin can compensate for all interim losses, but also on the risk of a shortfall in the early contract years. It is a perfect synthesis to overcome the disadvantages of the too short time horizon of Solvency II and too long time horizon from a regulatory point of view, as considered in the publications on the risk for life insurers due to the guarantees embedded in their variable annuities products. Moreover, it complies with the current U.S. RBC standard for variable annuities.

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Appendix 1

p(x, t) is a Solution of the Diffusion Equation, Which Vanishes on the Equity x=0-axis

Besides the function p_+ in equation (1), the Gaussian bell curve considered in the simple model, there are many other solutions of the diffusion equation (10)—for example, one for each starting point. For any initial equity c , without restriction on positive values for c , the function $\varphi_{c+mt, \sigma\sqrt{t}}(x)$ represents such a solution. With different values for c , the whole diffusion process will be translated only up or down on the x-axis. Let us now consider $c = -d$ —thus, the function $\varphi_{-d+mt, \sigma\sqrt{t}}(x)$. With a vanishing margin m , the two functions $\varphi_{d+mt, \sigma\sqrt{t}}(x)$ and $\varphi_{-d+mt, \sigma\sqrt{t}}(x)$ are mirror images of each other with regard to the equity-zero axis, and the difference of these two functions (11) and (12) vanishes on this equity-zero axis. In the case of a nonzero margin d , one has to introduce a constant coefficient multiplying one of these functions, cf. equation (13) below. To determine this coefficient, we compare the values of the two functions in question on the equity-zero axis

$$\varphi_{d+mt, \sigma\sqrt{t}}(0) = \eta \cdot e^{-\frac{1}{2} \frac{(d-mt)^2}{\sigma^2 t}} = \eta \cdot e^{-\frac{1}{2} \frac{d^2 + 2mt + (mt)^2}{\sigma^2 t}} \quad \text{and} \quad (11)$$

$$\varphi_{-d+mt, \sigma\sqrt{t}}(0) = \eta \cdot e^{-\frac{1}{2} \frac{(d-mt)^2}{\sigma^2 t}} = \eta \cdot e^{-\frac{1}{2} \frac{d^2 - 2mt + (mt)^2}{\sigma^2 t}} \quad (12)$$

The only difference between equation (11) and equation (12) consists of the mixed term

$$e^{-\frac{1}{2} \frac{4dm}{\sigma^2 t}} = e^{-\frac{2dm}{\sigma^2}}, \quad (13)$$

Thus, the time coordinate drops out, and the quotient (13) of the functions (11) and (12) is constant along the equity-zero axis for a given parameter set defining the specific Brownian motion. Hence,

$$\varphi_{d+mt, \sigma\sqrt{t}}(0) - e^{-\frac{2dm}{\sigma^2}} \varphi_{-d+mt, \sigma\sqrt{t}}(0) = 0 \quad \text{for } t > 0 \quad (14)$$

and the function

$$p = p(x, t) = p_+(x, t) - p_-(x, t) = \varphi_{d+mt, \sigma\sqrt{t}}(x) - e^{-\frac{2dm}{\sigma^2}} \varphi_{-d+mt, \sigma\sqrt{t}}(x) \quad (15)$$

vanishes on the equity-zero axis—i.e., on the line $(x, t) = (0, t), t > 0$. Therefore, the probability of a vanishing equity is zero for $p(x, t)$. That must be the case;

remember that $p(x, t)$ represents the probability density for an equity x at time t , whereby the equity has moved on a stochastic path during the period 0 to t without ever crossing the zero line.

A linear combination of solutions to a homogeneous differential equation is still a solution, and this is the case for the function p . Therefore, p is a solution to the diffusion equation (10), which describes the evolution of the equity of the insurance company based on the estimated volatility and margin appropriate to its business model and starting with its initial equity $d > 0$.

Appendix 2

Representing $\psi(t)/\psi(\infty)$ as Inverse Gaussian Distribution $IG_{\mu_{IG},\lambda}$

The following calculations are not required to understand the present article. They are given here, since the complex methods commonly used in actuarial literature first yield the inverse Gaussian distribution in the form given below, which must then be converted into the form used here as the sum of two cumulative normal distributions.

This representation yields to the expected time of ruin as parameter μ_{IG} . First, we look at the following question: If ruin occurs, at what time will it occur? Therefore, it is necessary to calculate the probability density that ruin occurs at a given time, assuming that it occurs. In the language of processes, this corresponds to the probability that the process path crosses the zero line of ruin until time t , if the path crosses it at all. Thus,

$$\psi(t)/\psi(\infty) = e^{-\frac{2md}{\sigma^2}} \Phi\left(-\frac{d+mt}{\sigma\sqrt{t}}\right) + \Phi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right)$$

Setting $g(t) = -\frac{d+mt}{\sigma\sqrt{t}}$ or $\frac{-d+mt}{\sigma\sqrt{t}}$ and applying the chain rule $\frac{d\Phi(g(t))}{dt} =$

$\varphi(g(t)) \frac{dg(t)}{dt}$ with

$$\frac{d\Phi}{dt} = \varphi_{0,1} = \varphi \text{ being the standard normal distribution } \varphi(z) = \left(\frac{1}{2\pi}\right)^{1/2} \cdot e^{-z^2/2},$$

leads to

$$\begin{aligned} \frac{d(\psi(t)/\psi(\infty))}{dt} &= e^{-\frac{2md}{\sigma^2}} \varphi\left(-\frac{d+mt}{\sigma\sqrt{t}}\right) \left(-\frac{m}{\sigma\sqrt{t}} + \frac{1}{2} \frac{d+mt}{\sigma t^{3/2}}\right) + \varphi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right) \left(\frac{m}{\sigma\sqrt{t}} + \frac{1}{2} \frac{d-mt}{\sigma t^{3/2}}\right) \\ &= \varphi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right) \left(-\frac{m}{\sigma\sqrt{t}} + \frac{1}{2} \frac{d+mt}{\sigma t^{3/2}} + \frac{m}{\sigma\sqrt{t}} + \frac{1}{2} \frac{d-mt}{\sigma t^{3/2}}\right) = \frac{d}{\sigma t^{3/2}} \varphi\left(\frac{-d+mt}{\sigma\sqrt{t}}\right) \\ &= \left(\frac{d^2/\sigma^2}{2\pi t^3}\right)^{1/2} \cdot e^{-\frac{1}{2} \frac{(t-d/m)^2}{(\sigma/m)^2 t}} = \left(\frac{\lambda}{2\pi t^3}\right)^{1/2} \cdot e^{-\frac{1}{2} \frac{\lambda}{\mu_{IG}^2 t} (t-\mu_{IG})^2} = p(IG_{\mu_{IG},\lambda}), \end{aligned}$$

where $p(IG_{\mu_{IG},\lambda})$ denotes the probability density function of the Inverse Gaussian random variable with mean $\mu_{IG}=d/m$ and shape parameter $\lambda=d^2/\sigma^2$.

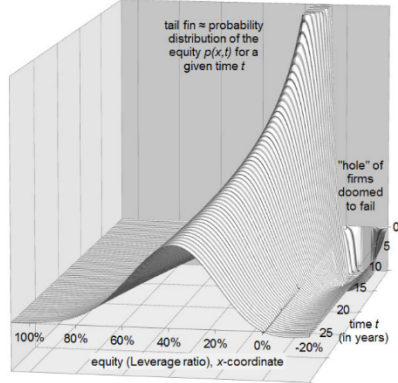
Appendix 3

The function $p(x, t)$ shown as surfaces

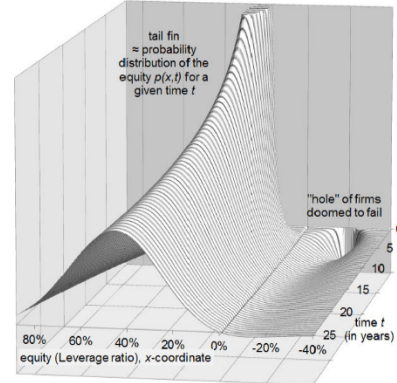
Figure 5 displays the whole graph of the function $p(x, t)$ as a two-dimensional surface in a three-dimensional space. It starts at the initial equity with a high probability density. The image resembles a tail fin, getting more and more broad as time passes by. The time coordinate is oriented from the background towards the viewer, with the known equity at the beginning of the process represented by a peak in the background.

Figure 5

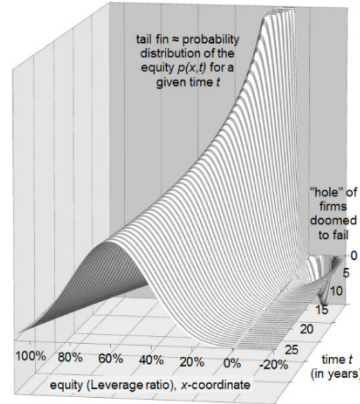
Ps 1: $d = 10\%$ and $m = 1\%$



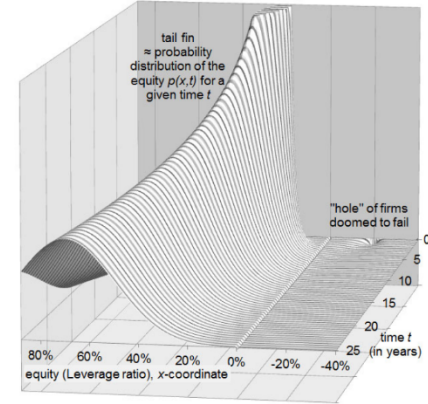
Ps 2: $d = 20\%$ and $m = 1\%$



Ps 3: $d = 10\%$ and $m = 2\%$



Ps 4: $d = 20\%$ and $m = 2\%$



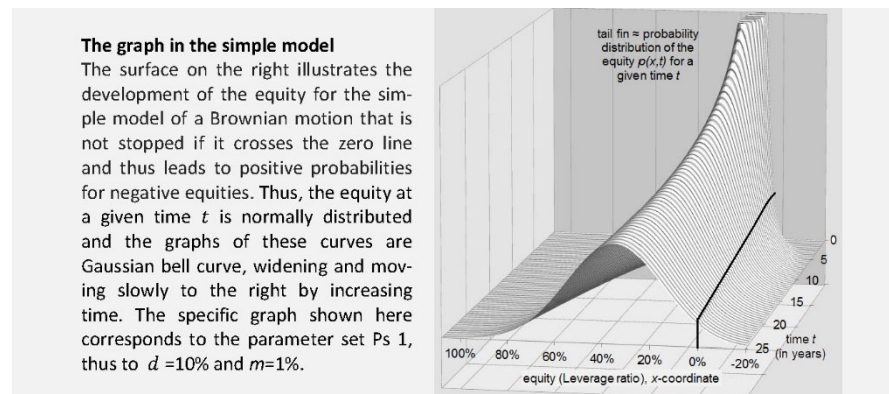
Since the process moves toward the foreground, the drift caused by the return margin drags the tail fin towards the left. The “red” line, that is the line where the process stops, is the time axis at zero percent equity.

Figure 5 illustrates that this kind of tail fin $p(x, t)$ crosses the zero level at equity $x = 0$ and then enters the range of negative equity values. This negative $p(x, t)$ on the right side—i.e., for negative x -coordinates from 0% to -20% and -40%, respectively—can be understood as the proportion of firms still active at the time but doomed to fail later on.

At the beginning of the process—i.e., for small values of time t —the negative part has approximately the size of $\psi(\infty)$. The higher the initial equity d at time $t = 0$ and the faster the process moves away from the stopping line $x = 0$ —that is, the higher return margin m —the smaller the size of the hole $\psi(\infty)$ needed to fulfill the imposed boundary condition on the stopping line. See Figure 5 Ps 4 for a small hole or Ps 1 for the converse case, where all images display the development over the next 25 years for all parameter sets, with time measured in years. In fact, the 25-year period depicted seems reasonable for a more holistic understanding of the risk situation and, therefore, provides a useful overview of the solvency situation.

For comparison, we have presented in Figure 6 the probability distributions of equity that result if the stochastic process is not stopped once the equity is depleted.

Figure 6



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